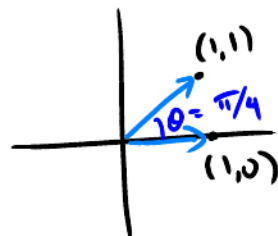
Last Time: Vectors and Geometry

↳ Cauchy-Schwarz Inequality

↳ Triangle Inequality

↳ $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

Ex: Compute angle between $(1,1)$ and $(1,0)$

Sol: $(1,1)\cdot(1,0) = 1 + 0 = 1$

$|(1,1)| = \sqrt{1^2+1^2} = \sqrt{2}$

$|(1,0)| = \sqrt{1^2+0^2} = 1$

$\therefore\; 1 = \sqrt{2}\cdot 1\cos(\theta)$ so $\cos(\theta) = \frac{1}{\sqrt{2}}$,

i.e. $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ ☑

Ex: Compute the angle between $\vec{u} = \begin{pmatrix}1\\0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\\pi\\1\\-5\end{pmatrix} = \vec{v}$.

Sol: $\vec{u}\cdot\vec{v} = -1 + 0 + 2 - 5 = -4$

$|\vec{u}| = \sqrt{1^2+0^2+2^2+1^2} = \sqrt{6}$

$|\vec{v}| = \sqrt{(-1)^2+\pi^2+1^2+(-5)^2} = \sqrt{27+\pi^2}$

$\therefore\; -4 = \sqrt{6}\sqrt{27+\pi^2}\cos(\theta)$ yields $\theta = \arccos\left(-\frac{4}{\sqrt{6}\sqrt{27+\pi^2}}\right)$.

Today: Reduced Row Echelon Form. (RREF)

Ex: Compute the RREF of $\left[\begin{array}{cccc|c} 0 & 1 & -1 & 2 & 3 \\ 2 & 4 & 0 & 0 & -6 \\ 2 & 5 & -1 & 2 & -3 \\ 3 & 1 & 4 & 1 & 5 \end{array}\right] = A$

Sol: Perform row operations:

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 2 & 3 \\ 2 & 4 & 0 & 0 & -6 \\ 2 & 5 & -1 & 2 & -3 \\ 3 & 1 & 4 & 1 & 5 \end{array}\right] \xrightarrow{\ell_2 \leftrightarrow \ell_1} \begin{bmatrix} 2 & 4 & 0 & 0 & -6 \\ 0 & 1 & -1 & 2 & 3 \\ 2 & 5 & -1 & 2 & -3 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}\ell_1} \begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 1 & -1 & 2 & 3 \\ 2 & 5 & -1 & 2 & -3 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix} \xrightarrow[\ell_4 - 3\ell_1]{\ell_3 - 2\ell_1} \begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & -5 & 4 & 1 & 14 \end{bmatrix}$$

← pivot entry (the circled 1 in row 1); pivot (the circled 1 in row 2)

$$\xrightarrow[\ell_3 - \ell_2,\ \ell_4 + 5\ell_2]{\ell_1 - 2\ell_2} \begin{bmatrix} 1 & 0 & 2 & -4 & -9 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 11 & 29 \end{bmatrix} \xrightarrow{\ell_3 \leftrightarrow \ell_4} \begin{bmatrix} 1 & 0 & 2 & -4 & -9 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & -1 & 11 & 29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\ell_3} \begin{bmatrix} 1 & 0 & 2 & -4 & -9 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -11 & -29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\ell_1 - 2\ell_3]{\ell_2 + \ell_3} \begin{bmatrix} 1 & 0 & 0 & 18 & 49 \\ 0 & 1 & 0 & -9 & -26 \\ 0 & 0 & 1 & -11 & -29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot entry (the circled 1 in row 3)

$$\therefore \ \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 18 & 49 \\ 0 & 1 & 0 & -9 & -26 \\ 0 & 0 & 1 & -11 & -29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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<u>Defn</u>: A matrix M is in <u>reduced row echelon form</u> (or simply RREF) when

1. All rows with only 0 entries are at the bottom of the matrix (i.e. every 0-row appears below every nonzero row).
2. Every nonzero row has a <u>leading 1</u> (i.e. the first nonzero entry in every nonzero row is 1)
3. Every leading 1 is the only nonzero entry

in its column.

④ Leading 1's appear in the same order left to right as they do top to bottom. (i.e. leftmost leading 1 is at top, etc).

Ex: Consider the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdot 1 \end{bmatrix}$$

this matrix IS in RREF!

□

Claim: Every matrix has a unique RREF

Def^n: Matrices A and B are row equivalent when there is a sequence of row operations transforming A into B.

Lem: Elementary row operations are reversible.

Elementary operations:
- Swap two rows,
- multiply a row by nonzero scalar.
- Add two rows, replace one.

pf: We treat each row operation separately:

Swaps: $\rho_i \leftrightarrow \rho_j$

is inverted by $\rho_i \leftrightarrow \rho_j$

$$i \to \begin{bmatrix} \rho_i \\ \rho_j \end{bmatrix} \leftarrow j \quad \leadsto \quad \begin{bmatrix} \rho_j \\ \rho_i \end{bmatrix}$$

Scaling: $k\rho_i$ is inverted by $\frac{1}{k}\rho_i$

(picture: $i\begin{bmatrix} \\ r_i \\ \\ \end{bmatrix} \xrightarrow{k\rho_i} \begin{bmatrix} \\ kr_i \\ \\ \end{bmatrix} \xrightarrow{\frac{1}{k}\rho_i} \begin{bmatrix} \\ \frac{1}{k}kr_i \\ \\ \end{bmatrix} = \begin{bmatrix} \\ r_i \\ \\ \end{bmatrix}$)

Add: $\rho_i + \rho_j \to \rho_j$ is inverted by the following sequence

$$M \xrightarrow{-\rho_i} M' \xrightarrow{\rho_i + \rho_j \to \rho_j} M'' \xrightarrow{-\rho_i} M'''$$

(Verify this using pictures :)) ☐

Point: Row equivalence is an equivalence relation:

① Every matrix is row equivalent to itself.

② If A is row equiv. to B, then B is row equivalent to A.

③ If A is row equiv. to B and B is row equiv. to C, then A is row equiv to C.

★ **Lem (Linear Combination Lemma):** A linear combination of linear combinations is itself a linear combination.

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i.e. Given linear combs:

$$\begin{aligned} a_{1,1}\vec{u}_1 + a_{1,2}\vec{u}_2 + \cdots + a_{1,n}\vec{u}_n &= \vec{v}_1 \\ a_{2,1}\vec{u}_1 + a_{2,2}\vec{u}_2 + \cdots + a_{2,n}\vec{u}_n &= \vec{v}_2 \\ &\vdots \\ a_{m,1}\vec{u}_1 + a_{m,2}\vec{u}_2 + \cdots + a_{m,n}\vec{u}_n &= \vec{v}_m \end{aligned}$$

Every linear combination of $\vec{v}_1, \cdots, \vec{v}_m$ is a linear combination of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$.

Pf: With the notation above, consider the linear combination of the v_i's below:

$$b_1 \vec{v}_1 + b_2 \vec{v}_2 + \cdots + b_m \vec{v}_m$$

$$\begin{aligned}
= \; & b_1 (a_{1,1} \vec{u}_1 + a_{1,2} \vec{u}_2 + \cdots + a_{1,n} \vec{u}_n) \\
+ \; & b_2 (a_{2,1} \vec{u}_1 + a_{2,2} \vec{u}_2 + \cdots + a_{2,n} \vec{u}_n) \\
& \vdots \\
+ \; & b_m (a_{m,1} \vec{u}_1 + a_{m,2} \vec{u}_2 + \cdots + a_{n,n} \vec{u}_n)
\end{aligned}$$

$$\begin{aligned}
= \; & b_1 a_{1,1} \vec{u}_1 + b_1 a_{1,2} \vec{u}_2 + \cdots + b_1 a_{1,n} \vec{u}_n \\
+ \; & b_2 a_{2,1} \vec{u}_1 + b_2 a_{2,2} \vec{u}_2 + \cdots + b_2 a_{2,n} \vec{u}_n \\
+ \; & \qquad \vdots \\
+ \; & b_m a_{m,1} \vec{u}_1 + b_m a_{m,2} \vec{u}_2 + \cdots + b_m a_{m,n} \vec{u}_n
\end{aligned}$$

$$\begin{aligned}
= \; & (b_1 a_{1,1} + b_2 a_{2,1} + \cdots + b_m a_{m,1}) \vec{u}_1 \\
+ \; & (b_1 a_{1,2} + b_2 a_{2,2} + \cdots + b_n a_{m,2}) \vec{u}_2 \\
& \qquad \vdots \\
+ \; & (b_1 a_{1,n} + b_2 a_{2,n} + \cdots + b_m a_{m,n}) \vec{u}_n
\end{aligned}$$

So the result is indeed a linear combination of $\vec{u}_i$'s ☑

Cor: If A is row equiv to B, then the rows of B are linear combinations of rows of A.

pf: We proceed by mathematical induction on the number of elementary operations performed to obtain B from A.

Base Case: If we perform 0 row operations, we have the same matrix.

So $\ell_1 = \ell_1$, $\ell_2 = \ell_2, \ldots, \ell_m = \ell_m$ are linear combinations of the old rows.

Induction step: Assume this holds for any sequence of n elementary row operations.

Applying one more row operation yields a linear combination of the resulting linear combinations (i.e. from the first n steps), because each row operation results in a linear combination of rows.

Hence, by the linear combination lemma, the result is a linear combination of rows of A.

By mathematical induction, the result holds ☑

Next Time: Finish the proof, and discuss consequences of uniqueness :)